# On the geometry of unbounded wandering domains

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# What is complex dynamics?

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We divide the complex plane $\mathbb{C}$ into the stable **Fatou set**

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The Julia set $\mathcal{J}$ divides the plane into the connected components of $\mathcal{F}$, which we call the **Fatou components**.

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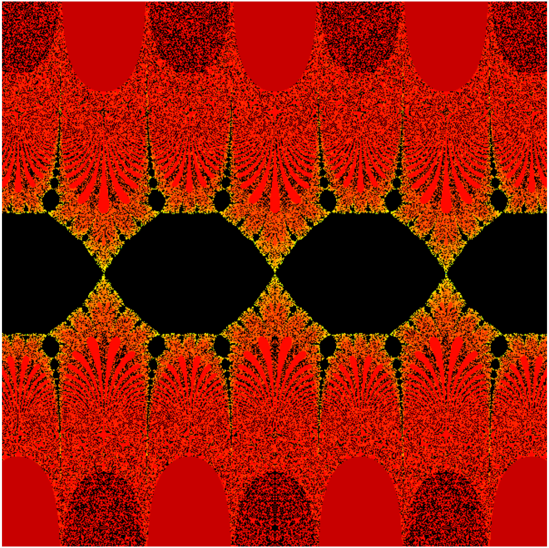
| **Periodic components** | **Wandering domains** |
|---|---|
| ▪ attracting basin<br>▪ parabolic basin<br>▪ Siegel disc<br>▪ Baker domain | ▪ escaping<br>▪ oscillating<br>▪ bounded |

# Example: The map $z \to z + \sin(z) + 2\pi$



# Geometry of wandering domains

**Theorem:** Boc Thaler (2021), Martí-Pete, Rempe, Waterman (2025)

Let $K \subseteq \mathbb{C}$ be a compact set with connected complement. Then there exists an entire function $f \in \mathcal{O}(\mathbb{C})$ for which every component of $\mathring{K}$ is an escaping wandering domain.

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Denote the strip $S = [-1, 1] \times \mathbb{R} \subseteq \mathbb{C}$ and let $F \subseteq S$ be a closed set that also satisfies some *reasonable* assumptions. Then there exists an entire function $f \in \mathcal{O}(\mathbb{C})$ for which every component of $\mathring{F}$ is an escaping wandering domain.

# Prescribing the geometry approximately

**Theorem: U.**

Let $F \subseteq U \subsetneq \mathbb{C}$, where $F$ is a closed and $U$ an unbounded simply connected open set. Then there exists an entire function $f \in \mathcal{O}(\mathbb{C})$ with an escaping wandering domain $\Omega$ such that $F \subseteq \Omega \subseteq U$.

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**Corollary: U.**

For each $\delta > 0$ there exists an entire function $f \in \mathcal{O}(\mathbb{C})$ with an escaping wandering domain $\Omega$ such that $\mathrm{Area}(\mathbb{C} \setminus \Omega) < \delta$.

# The tools of the trade

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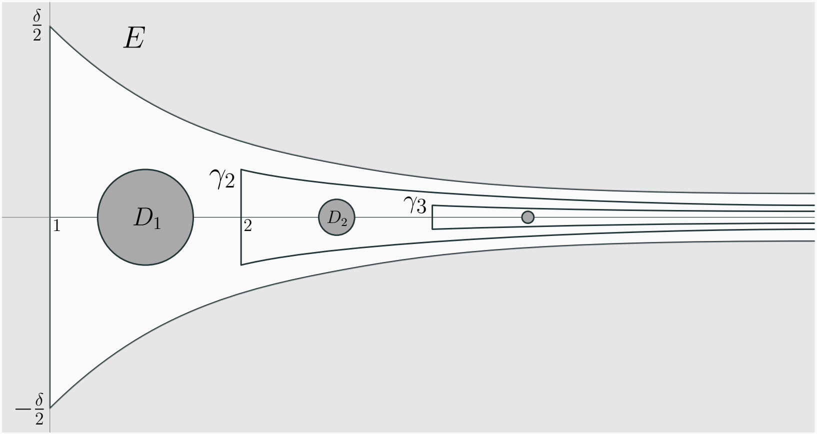
Let $F$ be closed and $F \subseteq U$ open and simply connected.
Then there exists an Arakelian set $E$ such that $F \Subset E \subseteq U$.

# Proof of the Corollary



# Proof of the Theorem

**Outline of the proof:**

1. Choose an Arakelian set $E$ such that $F \Subset E \subseteq U$.
2. Construct an Arakelian set $\Gamma \subseteq U \setminus E$ such that any component of $\mathbb{C} \setminus \Gamma$ intersects either $E$ or $\mathbb{C} \setminus U$. (Here we need that $U$ is unbounded.)
3. Construct curves $(\gamma_k)_{k \geq 1}$ and discs $(D_k)_{k \geq 1}$ in the component of $\mathbb{C} \setminus \Gamma$ that intersects $\mathbb{C} \setminus U$.
4. Use Arakelian approximation to obtain the map $f$.

**Thank you!**