Tame families of Stein Manifolds

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Joint work with Franc Forstnerič

Based on the paper *The Oka principle for tame families of Stein manifolds*

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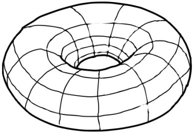
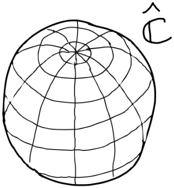
Compact manifolds are not Stein;





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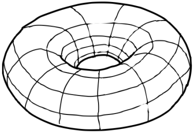

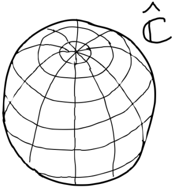

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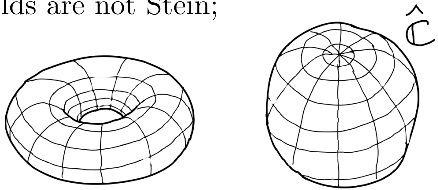

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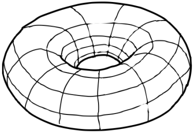

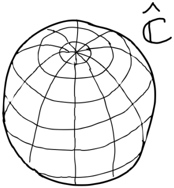

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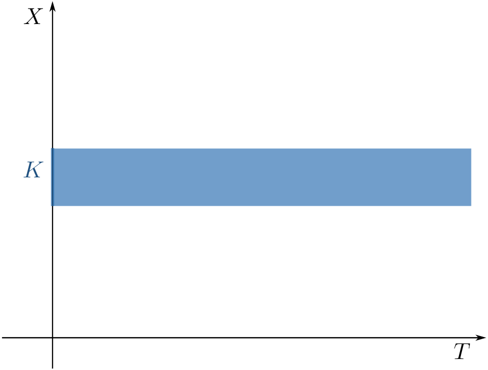
Oka-Weil theorem

If X is a Stein manifold, $K \subset X$ is compact and f is holomorphic in a neighbourhood of $\widehat{K}$, then we can approximate f uniformly on K with some holomorphic $F \colon X \to \mathbb{C}$.

Setup:

- X is a smooth manifold
- $\mathscr{J} = \{J_t\}_{t \in T}$ is a $\mathcal{C}^\infty$-continuous family of Stein structures.



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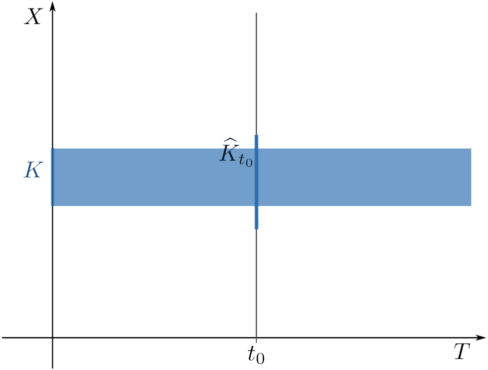

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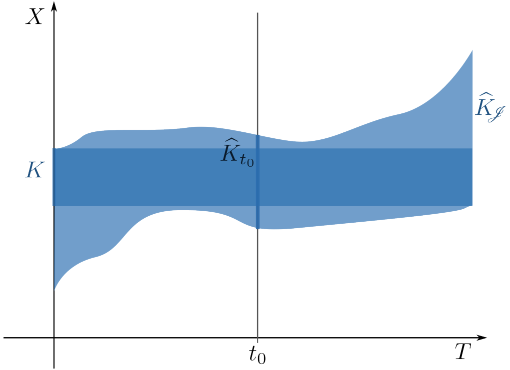

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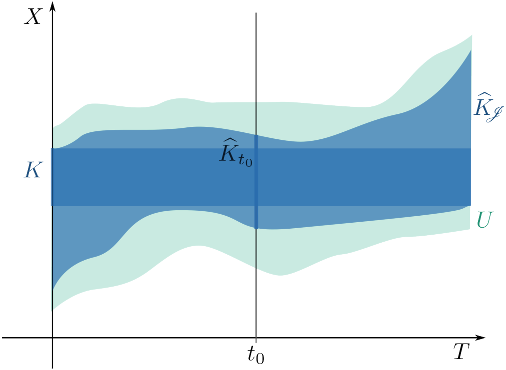

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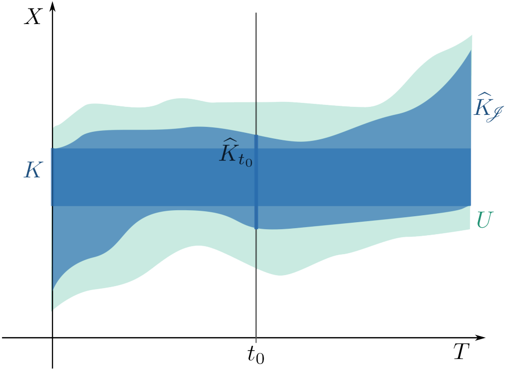

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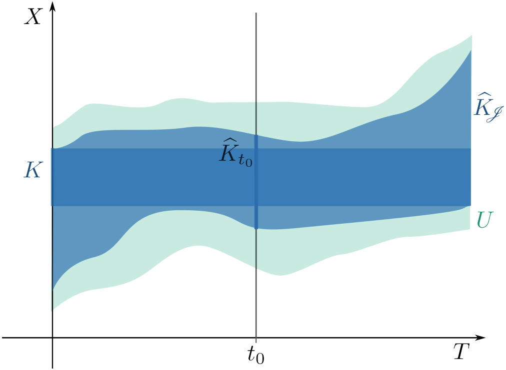

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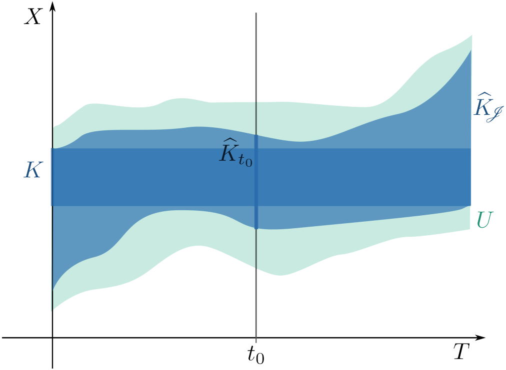

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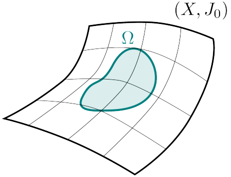

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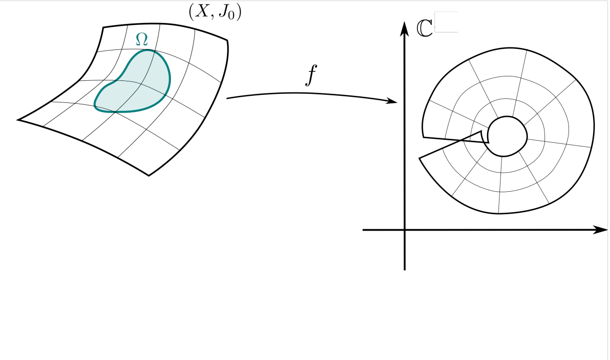

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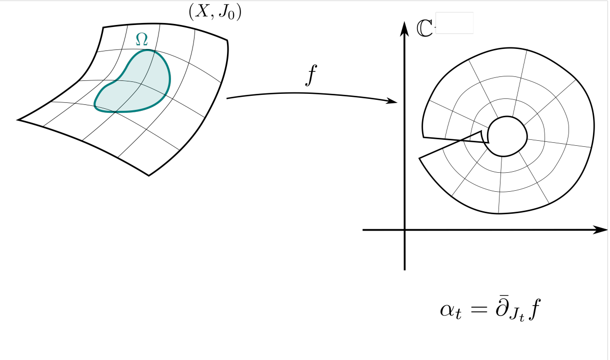

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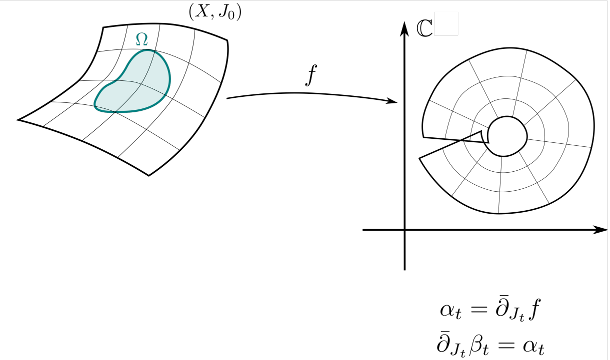

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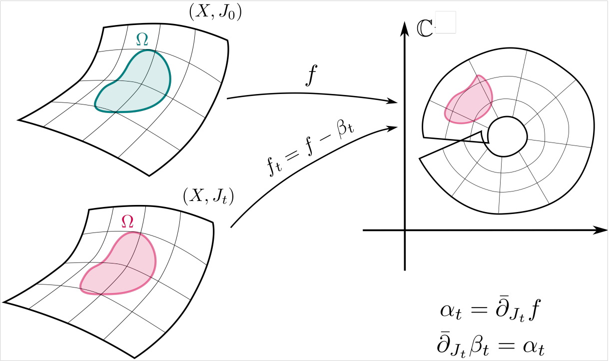

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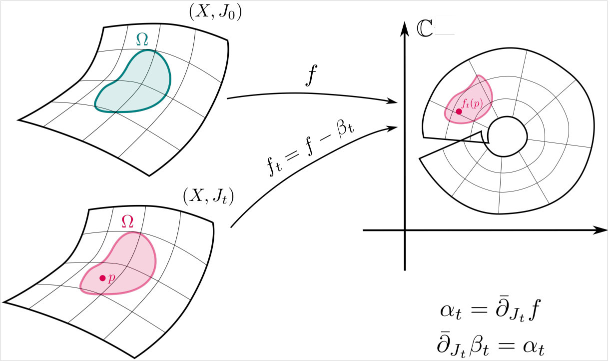

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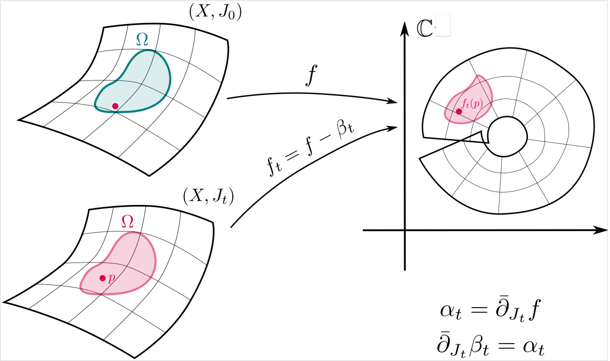

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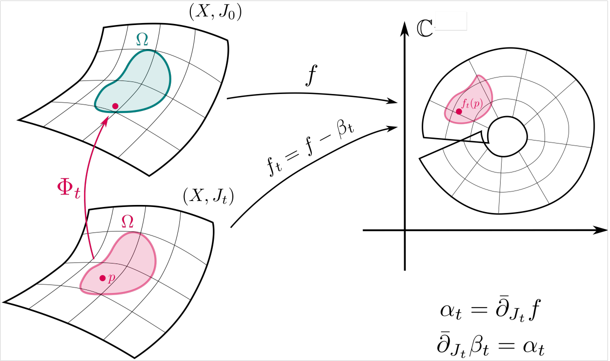

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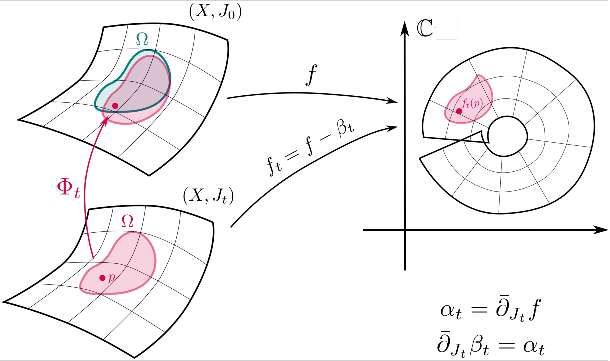

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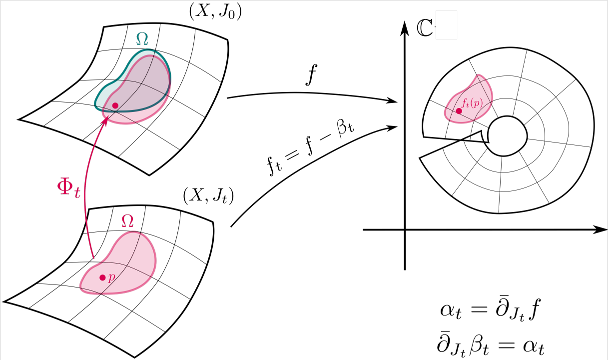
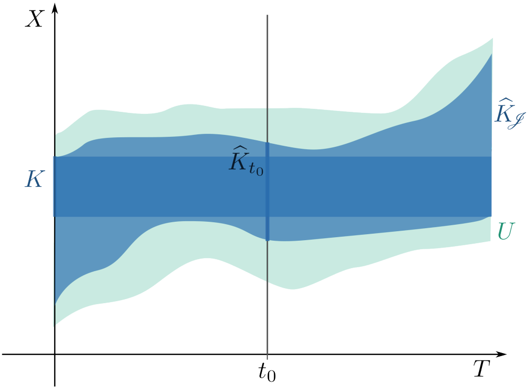
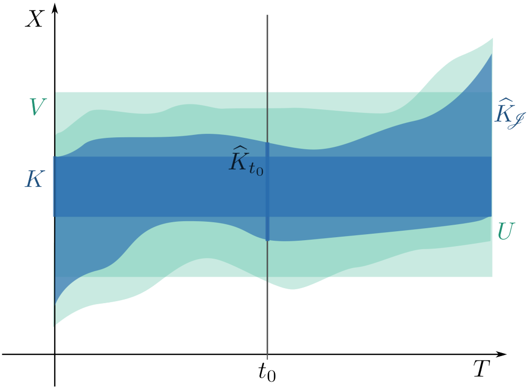


Theorem (Forstnerič & S. 2025)

If $\Omega \Subset X$ and $t_0 \in T$, there is a nbhd. T_0 of t_0 and a family $\Phi\colon T_0 \times \Omega \to X$ such that $\Phi_{t_0} = \mathrm{id}_\Omega$ and $\Phi_t : \Omega \to \Phi_t(\Omega) \subset X$ is a (J_t, J_{t_0})-biholomorphism depending continuously on $t \in T_0$.



X
K
$\widehat{K}_{t_0}$
$\widehat{K}_{\mathscr{J}}$
U
t_0
T



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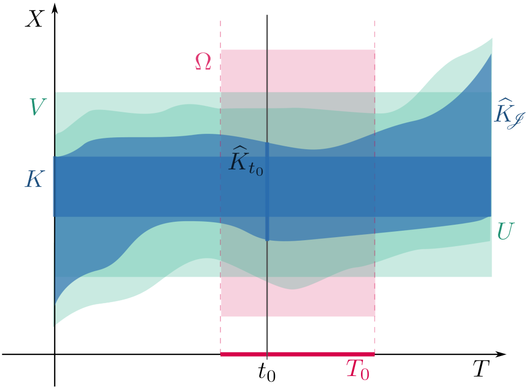

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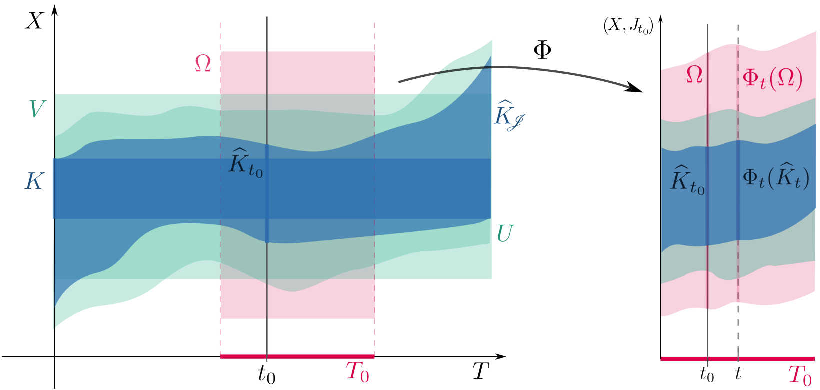

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t
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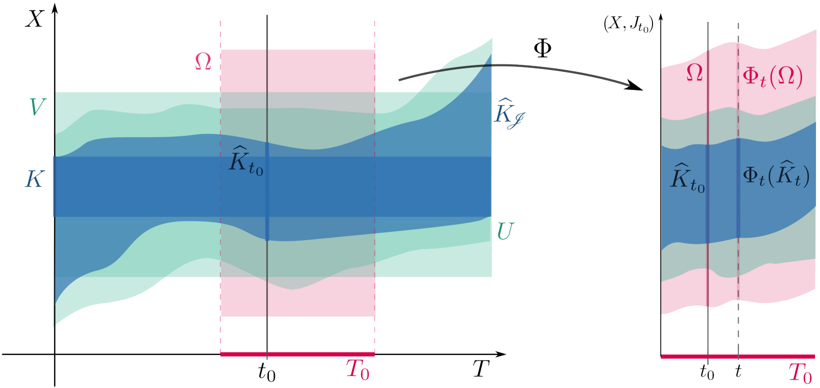

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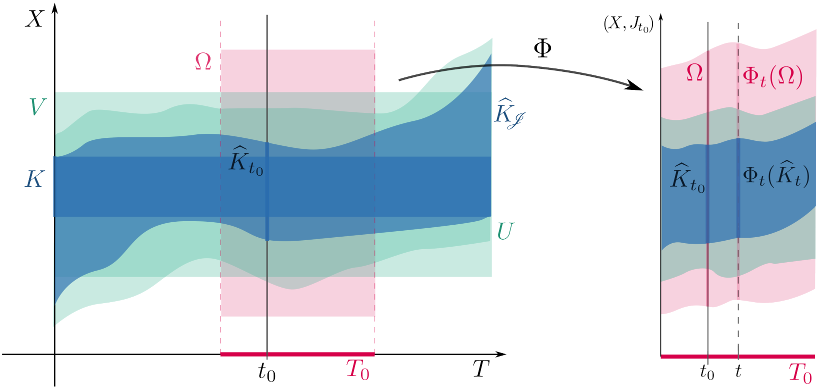

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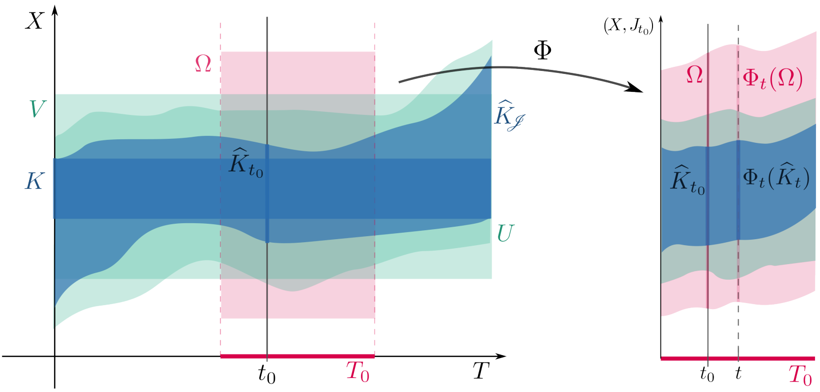

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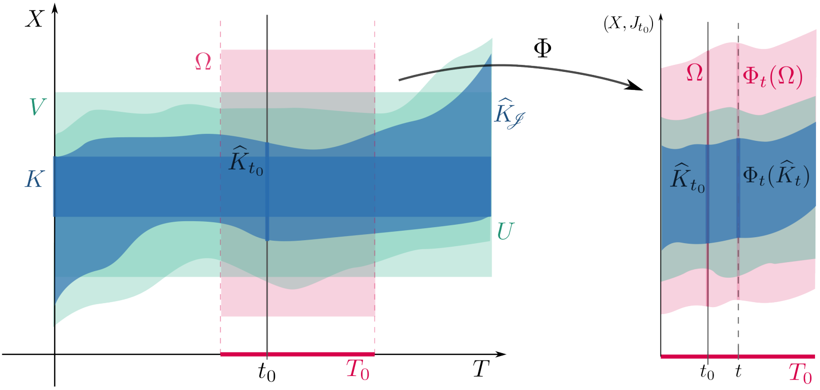

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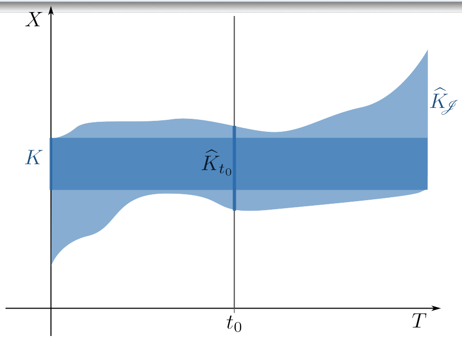
- P.o.u. $\{\chi_j\}_{j\in I}$ in T and take $F_t = \sum_{j\in I} \chi_j(t) F_t^j$.

Definition

A continuous family $\mathscr{J} = \{J_t\}_{t\in T}$ of complex structures on X is *tame* if for every compact $K \subset X$, its $\mathscr{J}$*-convex hull* $\widehat{K}_{\mathscr{J}} = \bigcup_{t\in T} \{t\} \times \widehat{K}_t$ is *locally bounded:*



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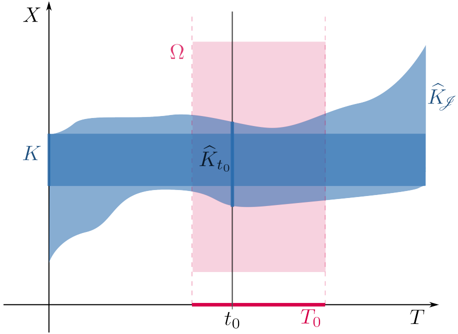

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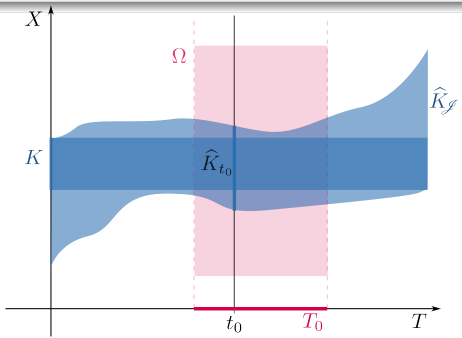

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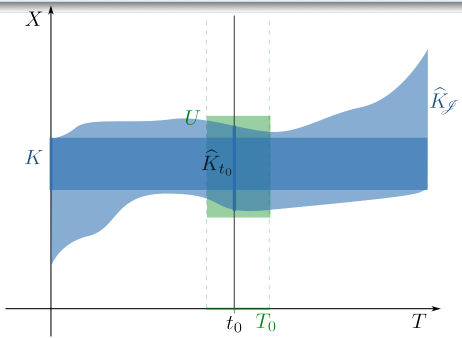

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i.e. if for any t_0 and an open set $U \subset X$ containing $\widehat{K}_{t_0}$ there is a neighbourhood $T_0 \subset T$ of t_0 such that $\widehat{K}_t \subset U$ for all $t \in T_0$.

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Given a continuous $\varepsilon\colon T \to \mathbb{R}^*_+$, there is a continuous $F\colon T \times X \to \mathbb{C}$ such that $F_t = F(t,\cdot)$ is J_t-holomorphic and $|F_t - f_t|_K < \varepsilon(t)$ for all $t \in T$.

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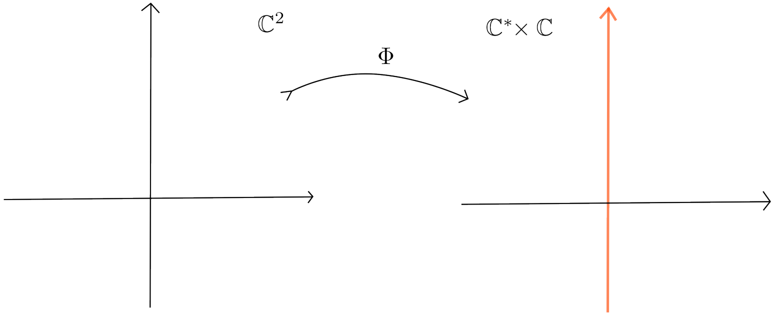
Idea: Wold's construction of an injective $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ whose image is not Runge.



Theorem (Forstnerič & S. 2025)

There exists a smooth family of Stein structures $\{J_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^4$ such that $J_0 = J_{\mathrm{std}}$, and for any neighbourhood $T_0 \subset \mathbb{R}$ of 0, the set $\bigcup_{t\in T_0} \widehat{K}_t \subset \mathbb{R}^4$ is unbounded, where K is the unit ball.

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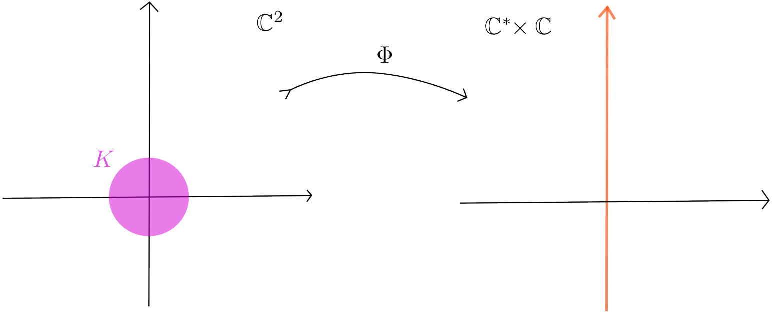

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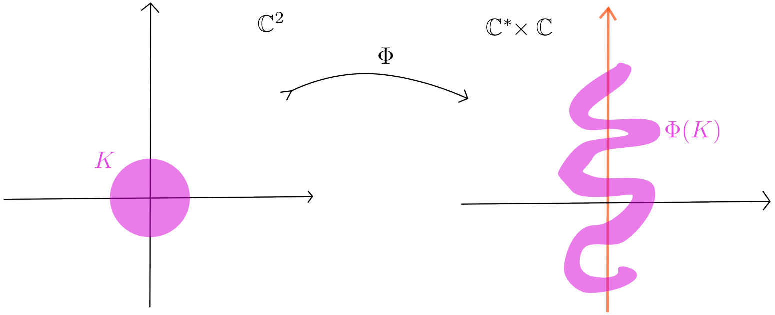

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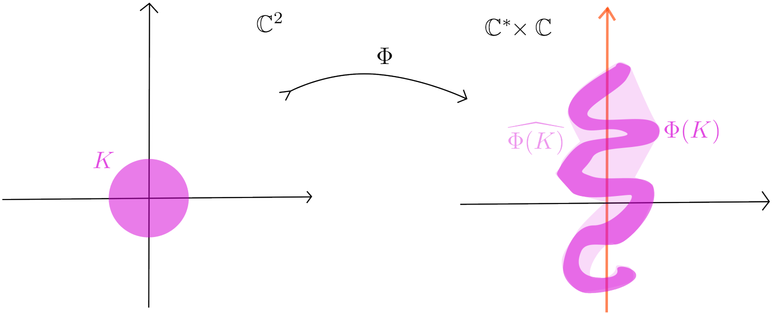

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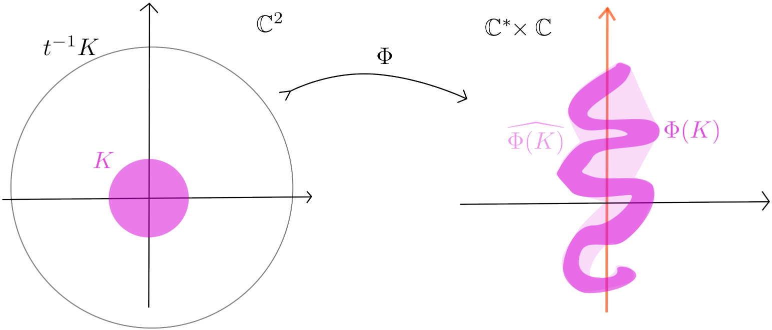

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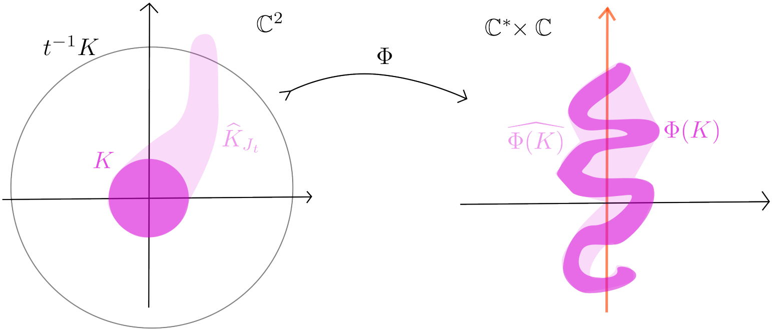

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Theorem (Forstnerič & S. 2025)

Assume that T is a finite CW complex, X is a smooth manifold, $\mathscr{J} = \{J_t\}_{t\in T}$ is a tame family of smooth Stein structures on X depending continuously on t, and Y is an Oka manifold.

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Then every continuous map $f : T \times X \to Y$ that is $\mathscr{J}$-holomorphic in a neighbourhood of $\widehat{K}_{\mathscr{J}}$ can be approximated on $\widehat{K}_{\mathscr{J}}$ by a $\mathscr{J}$-holomorphic map $F : T \times X \to Y$ that is homotopic to f.

Corollary

Assume that T is a locally compact and paracompact Hausdorff space, X is a smooth manifold, and $\mathscr{J} = \{J_t\}_{t\in T}$ is a continuous family of smooth Stein structures on X. The following are equivalent:

(i) If $\mathscr{J}$ is tame.

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(ii) For every $t_0 \in T$ and $f \in \mathscr{O}(X, J_{t_0})$, there is a $\mathscr{J}$-holomorphic $F : T \times X \to \mathbb{C}$ such that $F(t_0, \cdot) = f$.

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(iii) Let $p \geq 0$ and $q \geq 1$. Given a continuous family of smooth (p, q)-forms $\alpha_t \in \mathscr{E}^{p,q}(X, J_t)$ with $\bar{\partial}_{J_t}\alpha_t = 0$ for all $t \in T$, there exists a continuous family of $\beta_t \in \mathscr{E}^{p,q-1}(X, J_t)$ satisfying

$$\bar{\partial}_{J_t}\beta_t = \alpha_t \text{ on } X \text{ for every } t \in T.$$

and such that $\beta_t = 0$ whenever $\alpha_t = 0$.

Corollary

Assume that T *is a locally compact and paracompact Hausdorff space,* X *is a smooth manifold, and* $\mathscr{J} = \{J_t\}_{t \in T}$ *is a continuous family of smooth Stein structures on* X*. The following are equivalent:*

(i) *If* $\mathscr{J}$ *is tame.*

(ii) *For every* $t_0 \in T$ *and* $f \in \mathscr{O}(X, J_{t_0})$*, there is a* $\mathscr{J}$*-holomorphic* $F : T \times X \to \mathbb{C}$ *such that* $F(t_0, \cdot) = f$*.*

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Kiitos!