Brück's conjecture via complex linear DE:s

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A joint project with El Farissi, Dida and Zemirni

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Rainer Brück (born 1955) studied in Giessen under the supervision of Dieter Gaier, was scientifically active between 1986 – 2003, and is presently retired as a professor at TU Dortmund. His most cited paper is "*On entire functions which share one value CM with the first derivative*"in Results Math. 30 (1996), introducing what is called **Brück's conjecture** today.

In this talk, I assume that the audience is familiar with the standard notions and results of the Nevanlinna theory (for meromorphic functions), including the **proximity** function, **counting** function and **characteristic** function defined as follows:

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi}|d\phi$$

$$N(r,f) := \int_0^r \frac{n(t,\infty)-n(0,\infty)}{t}dt + n(0,\infty)\log r,$$

where $n(t,\infty)$ stands for the number of poles of f in the disc $|z|\leq t$, counting multiplicity.

$$T(r,f) := m(r,f) + N(r,f)$$

Moreover, we need to apply **the first main theorem**:

$$T(r,Af+B) = T(r,f) + O(1) \qquad T(r,1/f) = T(r,f) + O(1)$$

Before proceeding, recall that a meromorphic function g is said to be **small** with respect to another meromorphic function f provided $T(r, g) = S(r, f) = o(T(r, f))$ outside of a possible exceptional set of finite linear (resp. logarithmic) measure. Moreover, we need the notion CM, resp. IM: We say that two meromorphic functions f, g **share** an extended complex value α CM, resp. IM, if $f(z) = \alpha$ iff $g(z) = \alpha$ counting multiplicity, resp. ignoring multiplicity. One may also consider sharing a complex function $\alpha(z)$.

Brück's conjecture: Suppose f is a non-constant entire function with hyper-order

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} \notin \mathbb{N} \cup \{\infty\}.$$

If f and f' share a finite value α CM, then there exists a complex value $c \neq 0$ such that

$$\frac{f' - \alpha}{f - \alpha} = c.$$

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As far as I know, this conjecture remains open in its full generality. However, it has been proved in several special cases:

(1) Brück himself proved his conjecture, if $\alpha = 0$ or if $\alpha \neq 0$ with $N(r, 1/f') = S(r, f)$.

(2) Gundersen and Yang proved the conjecture (JMAA 1998), whenever f is of finite order,

(3) Chen and Shon proved the conjecture (Taiwanese J. Math. 2004), when $\rho_2(f) < 1/2$.

(4) Lahiri and Das recently proved the conjecture (2021), provided the lower hyper-order $\mu_2(f) := \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r} < 1/2$ and $\rho_2(f) < \infty$.

(5) An excellent survey by Lahiri (2020) describes a large number of results related to Brück. This survey also covers a number of Brück type results where $f, f^{(k)}$ had been considered instead of f, f' only.

Before proceeding, let me shortly comment proofs of the results mentioned here. The Brück case $\alpha = 0$ is essentially trivial. The Brück case with $N(r, 1/f') = S(r, f)$ is more complicated, making use of the Nevanlinna's **Second Main Theorem**, while the Gundersen–Yang case (2) makes use of Gundersen's pointwise estimates for the logarithmic derivatives. The Chen–Shon case (3) needs Wiman-Valiron type reasoning as well as the Lahiri–Das case (4). In this case (4), α may also be a polynomial instead of being a constant.

To obtain results extending (1) - (4), one has to look at the case when $\rho_2(f) \in [1/2, \infty]$. Moreover, a natural idea to proceed further would be to considering the situation when f, f' share a small (meromorphic) function $\alpha(z)$ CM instead of sharing a constant α.

Observe that the CM-sharing a meromorphic function α implies

$$\frac{f' - \alpha}{f - \alpha} = e^g,$$

where g is an entire function, since there are no zeros neither poles by the CM-sharing. This will be typically written as

$$f' - e^g f = (1 - e^g)\alpha.$$

As it seems to us, some **additional** idea is now perhaps needed to forward our knowledge around this Brück problem essentially.

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In a recent paper (Mediterr. J. Math. 2024), Dida and El Farissi propose to combine the Brück conjecture with an additional differential equation. In fact, they prove

Theorem

Let f be an entire function with $\rho_2(f) \in (1, \infty) \setminus \mathbb{N}$, and assume that f solves a second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0$$

with meromorphic coefficients A, B being of finite order. Then f, f' cannot share a finite value α CM.

Now, every entire function appears as a solution of $f'' + A(z)f' + B(z)f = 0$ with meromorphic coefficients A, B, and if all zeros of f, if any, are simple, then A, B are entire, see (CDE and Nikolaus in Math. Z. 1968). However, an entire function f with $\rho_2(f) \in (1, \infty) \setminus \mathbb{N}$ may fail to be a solution of $f'' + A(z)f' + B(z)f = 0$ with meromorphic coefficients of *finite order.* Indeed, given the entire functions

$$g_1(z) = \cos\left(z^2\sqrt{z}\right), \quad g_2(z) = e^z \quad \text{and} \quad g_3(z) = \frac{g_1(z) + g_2(z)}{2},$$

define $f(z) := e^{g_1(z)} + e^{g_2(z)} + e^{g_3(z)}$. For $r > 0$, we have $\log M(r, f) \asymp \exp\left(r^{5/2}\right)$, as $r \to +\infty$, and then $\rho_2(f) = \dfrac{5}{2} \in (1, +\infty) \setminus \mathbb{N}$.

Assume that $f(z) := e^{g_1(z)} + e^{g_2(z)} + e^{g_3(z)}$ is a solution of $f'' + A(z)f' + B(z)f = 0$ with meromorphic coefficients of finite order. Then

$$\sum_{j=1}^{3} \left[\left(g_j'\right)^2 + g_j'' + Ag_j' + B \right] e^{g_j} = 0.$$

By a standard lemma, we conclude that the coefficients vanish:

$$\begin{cases} (g_1')^2 + g_1'' + Ag_1' + B = 0 \\ (g_2')^2 + g_2'' + Ag_2' + B = 0 \\ (g_3')^2 + g_3'' + Ag_3' + B = 0 \end{cases}$$

To prove the vanishing of these coefficients, the following typical lemma is needed (see e.g. the book by Yang and Yi):

Lemma

Let $n \geq 2$. Suppose that $f_1, f_2, \ldots, f_n, f_{n+1}$ are meromorphic functions and $g_1, g_2, \ldots, g_n$ are entire functions satisfying :

1. $f_1 e^{g_1} + \cdots + f_n e^{g_n} = f_{n+1}$.
2. *g_s and $g_j - g_k$ are non-constant for every $1 \leq s \leq n$ and $1 \leq j < k \leq n$.*
3. *For $1 \leq \ell \leq n+1$, $1 \leq j < k \leq n$, and $1 \leq s \leq n$, we have*

$$T(r, f_\ell) = S(r, e^{g_j - g_k}) \quad \text{and} \quad T(r, f_\ell) = S(r, e^{g_s}).$$

Then $f_j(z) \equiv 0$ for every $j = 1, 2, \cdots, n+1$.

Recalling now that $g_3 = (g_1 + g_2)/2$, the above system then yields

$$(\cos(z^2\sqrt{z}))' = g_1' = g_2' = (e^z)',$$

which is not possible. Thus f cannot be a solution of $f'' + A(z)f' + B(z)f = 0$ with meromorphic coefficients of finite order.

However, f can still be a solution to

$$f'' + A(z)f' + B(z)f = F(z),$$

where F is meromorphic non-vanishing and of finite order. In fact, we may consider the system

$$\begin{cases} (g_1')^2 + g_1'' + Ag_1' + B = 0 \\ (g_2')^2 + g_2'' + Ag_2' + B = 0 \\ (g_3')^2 + g_3'' + Ag_3' + B = F \end{cases}.$$

By replacing g_3 with $(g_1 + g_2)/2$, this system yields

$$F(z) = \frac{g_1' g_2' - (g_1')^2 - (g_2')^2}{4} = -\frac{5}{2}\left(e^z + \frac{5}{2} z\sqrt{z}\sin\left(z^2\sqrt{z}\right)\right) z\sqrt{z}\sin (z^2\sqrt{z}$$

Since $g_1' - g_2' \neq 0$, the functions A and B are now given by

$$\begin{bmatrix} A \\ B \end{bmatrix} = -\begin{bmatrix} g_1' & 1 \\ g_2' & 1 \end{bmatrix}^{-1} \begin{bmatrix} (g_1')^2 + g_1'' \\ (g_2')^2 + g_2''. \end{bmatrix}$$

Returning back to Nikolaus, and making use of repeated differentiation, it becomes clear to obtain the following

Theorem

An arbitrary entire function f is a solution of a differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$$

for any order $k \geq 2$ with meromorphic coefficients.

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Let me now state the key result of my talk:

Theorem

Let f be an entire function with $\rho_2(f) \in (1, \infty) \setminus \mathbb{N}$, and assume that f is a solution of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \qquad k \geq 2$$

with meromorphic coefficients of finite order. Then f, f' cannot share an entire function $\alpha(z)$ of finite order CM.

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To start with, we must have

$$\frac{f'(z) - \alpha(z)}{f(z) - \alpha(z)} = e^{g(z)},$$

where $g(z)$ is entire, since by CM the quotient here has no zeros nor poles.

First note that if g is constant, say d, then

$$f(z) = Ce^{d} + e^{dz} \int_{1}^{z} (1 - d)e^{-dt}\alpha(t)dt$$

meaning that f is of finite order, contradicting assumption that $\rho_2(f) > 1$. Therefore, we may now assume that g is non-constant.

Recall now, that f solves two differential equations at the same time:

$$f'(z) - e^{g(z)} f(z) = (1 - e^{g(z)})\alpha(z)$$

and

$$f^{(k)}(z) + A_{k-1}(z) f^{(k-1)}(z) + \cdots + A_1(z) f'(z) + A_0(z) f(z) = 0$$

with meromorphic coefficients of finite order.

Differentiate now $f' - e^g f = (1 - e^g)\alpha$ and substitute this f' back into the differentiated expression to obtain

$$f'' = (e^{2g} + g' e^g)(f - \alpha) + (\alpha - \alpha')e^g + \alpha'.$$

Proceeding inductively, we obtain

$$f^{(s)} = \left(\sum_{j=1}^{s} H_{j,s} e^{jg} \right) (f - a) + \sum_{j=0}^{s-1} K_{j,s} e^{jg},$$

where $H_{j,s}$ are polynomials in g and its derivatives with constant coefficients, and where $K_{j,s}$ are polynomials in α, g and their derivatives with constant coefficients.

Constructing now expressions for $f', f'', \ldots, f^{(k)}$ and substituting them in $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$ we get

$$\left(\sum_{j=0}^{k} M_j e^{jg}\right)\left(\frac{f}{\alpha} - 1\right) = \left(\sum_{j=0}^{k-1} N_j e^{jg}\right),$$

where M_j are small meromorphic functions (with respect to e^g) as they depend on g and its derivatives, and depend on finite order coefficients $A_0, \ldots, A_{k-1}$, and N_j are small similarly, depending on g, α, their derivatives and on $A_0, \ldots, A_{k-1}$. In particular, $M_k = 1$.

Assume now, for a while, that g is a **transcendental entire function**, meaning that e^g is of *infinite order*, and proceed to a contradiction.

Next observe that $\sum_{j=0}^{k} M_j e^{jg}$ cannot vanish. Indeed, if so, Lemma 2 implies that the coefficients M_j vanish which is not possible. But then, dividing off possible common factors, we may write

$$\frac{f}{\alpha} - 1 = \frac{P(e^g)}{Q(e^g)} = \frac{\sum_{j=0}^{d-1} S_j e^{jg}}{\sum_{j=0}^{d} L_j e^{jg}},$$

where P, Q are relatively prime polynomials in e^g over the field of small meromorphic functions. The coefficients S_j, L_j are small with respect to e^g, resp. to f. In particular, $S_{d-1} = \frac{\alpha'}{\alpha} - 1$ and $L_d = 1$.

Invoking the Valiron–Mohon'ko theorem, we readily obtain

$$T(r,f) = T(r, f/\alpha - 1) + S(r,f) = T(r, P(e^g)/Q(e^g)) + S(r,f)$$

$$= dT(r, e^g) + S(r,f).$$

This now implies that $S(r,e^g) = S(r,f)$, meaning that a function is small for f and for e^g at the same time.

Denote now $F := \frac{f}{\alpha} - 1(= P(e^g)/Q(e^g))$. By $f' - e^g f = (1 - e^g)\alpha$, we see that

$$F' + (\frac{\alpha'}{\alpha} - e^g)F = 1 - \frac{\alpha'}{\alpha}.$$

Compute next F' to obtain

$$F' = \frac{(P(e^g))' Q(e^g) - (Q(e^g))' P(e^g)}{(Q(e^g))^2}.$$

Substituting now $F' = \frac{(P(e^g))'Q(e^g)-(Q(e^g))'P(e^g)}{(Q(e^g))^2}$ and $F = \frac{f}{\alpha} - 1 = P(e^g)/Q(e^g)$ into

$$F' + (\frac{\alpha'}{\alpha} - e^g)F = 1 - \frac{\alpha'}{\alpha},$$

we obtain

$$(P(e^g))'Q(e^g) - (Q(e^g))'P(e^g) + (\frac{\alpha'}{\alpha} - e^g)P(e^g)Q(e^g) = (1 - \frac{\alpha'}{\alpha})(Q(e^g))^2.$$

Write now

$$P(e^g)(Q(e^g))' = R(e^g)Q(e^g).$$

Recalling that $Q(e^g) = \sum_{j=0}^{d} L_j e^{jg}$ we see that

$$\tilde{Q}(e^g) := (Q(e^g))' = (\sum_{j=0}^{d} L_j e^{jg})' = \sum_{j=0}^{d} (jg' L_j + L_j')e^{jg}.$$

Now, it is clear that $\deg \tilde{Q} = d = \deg Q$. Therefore, we now obtain

$$P(e^g)\tilde{Q}(e^g) = R(e^g)Q(e^g).$$

Since P, Q are relatively prime, Q must divide $\tilde{Q}$. This means that

$$\tilde{Q}(e^g) = (Q(e^g))' = hQ(e^g),$$

where h is a meromorphic function small relative to f. Therefore,

$$\tilde{Q}(e^g) - hQ(e^g) = \sum_{j=0}^{d}(jg'L_j + L_j' - hL_j)e^{jg} = 0.$$

Again, we may apply Lemma 2 to see that the coefficients here vanish. In particular, the leading coefficient $dg'L_d + L_d' - hL_d$ vanishes, and so, since $L_d = 1$,

$$h = dg'.$$

Therefore,

$$(Q(e^g))' = hQ(e^g) = dg'Q(e^g)$$

and so

$$Q(e^g) = ce^{dg}(= \sum_{j=0}^{d} L_j e^{jg}).$$

Since $L_d = 1$, we see that $Q(e^g) = e^{dg}$.

Therefore, using $Q(e^g) = e^{dg}$, we may write

$$(P(e^g))'Q(e^g) - (Q(e^g))'P(e^g) + (\frac{\alpha'}{\alpha} - e^g)P(e^g)Q(e^g) = (1 - \frac{\alpha'}{\alpha})(Q(e^g))^2.$$

in short as

$$(P(e^g))' - \left(e^g + dg' - \frac{\alpha'}{\alpha}\right) P(e^g) = \left(1 - \frac{\alpha'}{\alpha}\right) e^{dg}.$$

Recalling that $P(e^g) = \sum_{j=0}^{d-1} S_j e^{jg}$, we obtain, by substitution,

$$S_0' - (dg' - \frac{\alpha'}{\alpha})S_0 + \sum_{j=1}^{d-1}(((j-d)g' + \frac{\alpha'}{\alpha})S_j + S_j' - S_{j-1})e^{jg} = 0.$$

By Lemma 2 again, the coefficients here must vanish. If now $S_0 \neq 0$, then $S_0 = ce^{dg}/\alpha$, since $S_0' - (dg' - \frac{\alpha'}{\alpha})S_0 = 0$. Since S_0 and α are small (relative to e^g), we have a contradiction.

Proceeding now (from $S_0 = 0$) to the first non-vanishing coefficient $S_m \neq 0, 1 \leq m \leq d-1$, a similar contradiction follows. In fact, since we now have $S_m \neq 0$ and $S_{m-1} = 0$, we must have $((m-d)g' + \frac{\alpha'}{\alpha})S_m + S_m' = 0$ and so a contradiction $S_m = ce^{(d-m)g/\alpha}$ follows. Since, $P(e^g), Q(e^g)$ are non-vanishing, such a non-vanishing S_m must exist. So, we get a semifinal contradiction, i.e. that g cannot be transcendental.

Therefore, g must be a polynomial. Clearly, we may assume that g is non-constant. Since $\rho_2(f) \in (1, \infty) \backslash \mathbb{N}$ is entire and the entire function $(1 - e^g)\alpha$ is small with respect to f, we may apply a result by Cao (Bull. Austr. 2016):

Theorem (Cao): Let P be a non-constant polynomial and let f be a nonzero entire solution of $f' - e^P f = Q$, where Q is an entire function that is 'small' with respect to f. Then $\rho_2(f) = \deg g$.

The notion of $'small'$ in Cao's theorem is unclear, however. But looking at the proof of this result in the paper by Cao, the key point there is that

$$|Q(z_r)|/M(|z_r|, f) = o(1)$$

must hold outside of an exceptional r-set of finite logarithmic measure. But now, in $Q = (1 - e^g)\alpha$, g is a polynomial and α is of finite order. Thus, $Q = (1 - e^g)\alpha$ is of finite order. Since $\rho_2(f) > 1$, this condition is being satisfied. Therefore, $\rho_2(f) = \deg g \in \mathbb{N}$, a **contradiction**.

What to do next ?

Recall our

Theorem

Let f be an entire function with $\rho_2(f) \in (1, \infty) \setminus \mathbb{N}$, and assume that f is a solution of

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \qquad k \geq 2$$

with meromorphic coefficients of finite order. Then f, f' cannot share an entire function $\alpha(z)$ of finite order CM.

What about:

1. Then f, f' cannot share a small entire function $\alpha(z)$ CM (instead of finite order α) ?? Small for f and e^g at the same time ??

2. Brück conjecture when $\rho_2(f) \in [1/2, 1)$? What about a non-homogeneous additional DE with finite order coefficients ? What about if $\rho_2(f) \in \mathbb{N}$?

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